Tutorial 4

Solving matrix games

Two useful principles: 1. Deleting the dominated rows and columns to obtain a new matrix with lower dimensions. Recall that a row is dominated if it is dominated (or say bounded) from above by another row, a column is dominated if it is dominated from below by another column.

2. The principle of indifference. Assume $\boldsymbol{p} = (p_1, \cdots, p_m)$ and $\boldsymbol{q} = (q_1, \cdots, q_n)$ are optimal strategies for Player I and Player II respectively. Then

(i) for any $k \in \{1, \cdots, m\}$ with $p_k > 0$, we have $\sum_{j=1}^{n} a_{k,j} q_j = v(A)$.

(ii) for any $l \in \{1, \cdots, n\}$ with $q_l > 0$, we have $\sum_{i=1}^{m} a_{i,l} p_i = v(A)$.

Exercise 1. *In a Rock-Paper-Scissors game, the loser pays the winner an amount of money which is equal to the total number of fingers shown by the two players (for example, if Player I shows Scissors and Player II shows Paper, then Player II should pay 7 dollars to Player I).*

(i) Find the value of the games.

(ii) Find optimal strategies for the two players.

Solution. The game is clearly a two-person zero-sum game and the game matrix is given by

$$A = \begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & -7 \\ -2 & 7 & 0 \end{pmatrix} \end{array}.$$

(i) Since $A^T = -A$, we have $v(A) = 0$.

(ii) Assume $\boldsymbol{q} = (q_1, q_2, q_3)$ is an optimal strategy for Player I. Assume

q_1, q_2, q_3 are all positive, then by the principle of indifference, we have

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & -7 \\ -2 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$\begin{cases} 5p_2 - 2p_3 = 0 \\ -5p_1 + 7p_3 = 0 \\ 2p_1 - 7p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Solving the above equations, we get $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{7}$, $p_3 = \frac{5}{14}$. Similarly, assume $\boldsymbol{p} = (p_1, p_2, p_3)$ is an optimal strategy for Player II and $\boldsymbol{p}$ is strictly positive, we have $\boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$. It is easy to check $v = 0$, $\boldsymbol{p} = \boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$ satisfy the the conclusion of the Minimax Theorem. Hence $v = 0$ is the value of A and $\boldsymbol{p} = \boldsymbol{q} = (\frac{1}{2}, \frac{1}{7}, \frac{5}{14})$ are optimal strategies.

Exercise 2. *Let*

$$A = \begin{pmatrix} 0 & -2 & 2 & 1 & 4 \\ 2 & -1 & 3 & 0 & 5 \\ 3 & 4 & -2 & 5 & -3 \end{pmatrix}$$

(i) Find the reduced matrix of A by deleting dominated rows and columns.

(ii) Solve the two-person zero-sum game with game matrix A.

Solution. (i) Note that the fourth column is dominated by the second column from below, by deleting the fourth column we obtain

$$\begin{pmatrix} 0 & -2 & 2 & 4 \\ 2 & -1 & 3 & 5 \\ 3 & 4 & -2 & -3 \end{pmatrix}.$$

Now the first row in dominated by the second row from above, by deleting the first row we obtain

$$\begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 4 & -2 & -3 \end{pmatrix}.$$

There are no more dominated rows or columns, hence the above matrix is the desired reduced matrix.

(ii) Let A' denote the reduced matrix. For $x \in [0,1]$, we have

$$(x, 1-x)A' = (2x + 3(1-x), -x + 4(1-x), 3x - 2(1-x), 5x - 3(1-x)).$$

Draw the graph of

$$\begin{cases} C_1 : v = 2x + 3(1-x) = 3 - x \\ C_2 : v = -x + 4(1-x) = 4 - 5x \\ C_3 : v = 3x - 2(1-x) = 5x - 2 \\ C_5 : v = 5x - 3(1-x) = 8x - 3 \end{cases}.$$

The lower envelope is shown in Figure 1. Solving

$$\begin{cases} C2 : v = 4 - 5x \\ C3 : v = 5x - 2 \end{cases},$$

we have $v = 1$ and $x = 0.6$. Hence $v(A) = 1$ and the optimal strategy for the row player is $(0, 0.6, 0.4)$. Solving

$$\begin{cases} R2 : -y + 3(1-y) = 1 \\ R3 : 4y - 2(1-y) = 1 \end{cases},$$

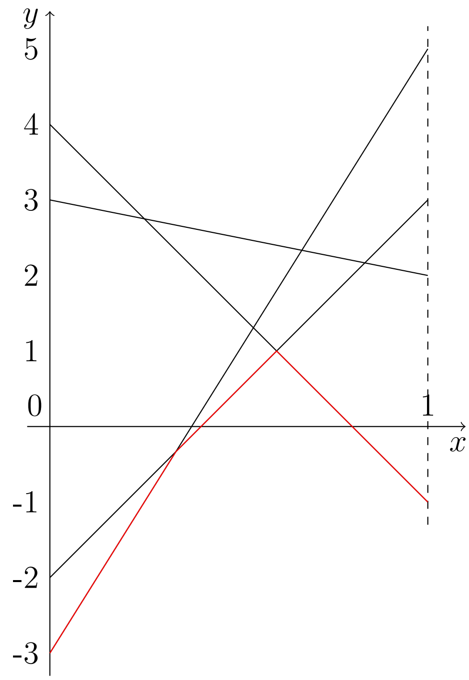

Figure 1:

we have $y = 0.5$. Hence the optimal strategy for the column player is $(0, 0.5, 0.5, 0, 0)$.

Recall the Minimax Theorem. This existence theorem also gives a characterization of the value of a game matrix and optimal strategies for the two players. More precisely, given an $m \times n$ matrix A, we call a number v the value of A, a probability vector $\boldsymbol{p} \in \mathcal{P}^m$ a maximin strategy for the row player, and a probability vector $\boldsymbol{q} \in \mathcal{P}^n$ a minimax strategy for the column player if

(i) $\boldsymbol{p}A\boldsymbol{y}^T \geq v$ for any $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x}A\boldsymbol{q}^T \leq v$ for any $\boldsymbol{x} \in \mathcal{P}^m$.

(iii) $\boldsymbol{p}A\boldsymbol{q}^T = v$.

We note condition (i) is equivalent to

(i)$'$ every element of the row vector $\boldsymbol{p}A$ is at least v, and the condition (ii) is equivalent to

(ii)$'$ every element of the column vector $A\boldsymbol{q}^T$ is at most v.

Exercise 3. *Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. Let M be the $(m+n) \times (m+n)$ matrix given by*

$$M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Let u be the value, $\boldsymbol{p} \in \mathcal{P}^m$ be a maximin strategy for the row player and $\boldsymbol{q} \in \mathcal{P}^m$ be a minimax strategy for the column player of A. Let v be the value, $\boldsymbol{r} \in \mathcal{P}^n$ be a maximin strategy for the row player and $\boldsymbol{s} \in \mathcal{P}^n$ be a minimax strategy for the column player of B.

(i) Suppose $u > 0$ and $v < 0$. Find the value of M and optimal strategies for the two players of the game with game matrix M.

(ii) Suppose $u > 0$ and $v > 0$. Find the value of M in terms of u and v. Find optimal strategies for the row player and the column player of M in terms of $u, v, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$.

Solution. (i) Note that

$$\begin{pmatrix} \boldsymbol{p} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}A & \boldsymbol{0} \end{pmatrix},$$

and

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}^T \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ B\boldsymbol{s}^T \end{pmatrix}.$$

Since $u > 0$ and $\boldsymbol{p}$ is an maximin strategy for the row player of A, we have every element of the $m+n$ dimensional row vector $(\boldsymbol{p}A, \boldsymbol{0})$ is at least 0. Similarly, since $v < 0$, we have every element of the $m+n$ dimensional column $(\mathbf{0}, \boldsymbol{s}B^T)^T$ is at most 0. Clearly,

$$\begin{pmatrix} \boldsymbol{p} & \mathbf{0} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}^T \end{pmatrix} = 0.$$

Hence by the Minimax Theorem, the value of M equals 0, $(\boldsymbol{p}, \mathbf{0})$ is a maximin strategy for the row of player of M and $(\mathbf{0}, \boldsymbol{s})$ is a minimax strategy for the column player of M.

(ii). In the case that $u, v > 0$, we start by assuming that for some $\lambda \in [0, 1]$ (to be determined), $(\lambda\boldsymbol{p}, (1-\lambda)\boldsymbol{r})$ and $(\lambda\boldsymbol{q}, (1-\lambda)\boldsymbol{s})$ are optimal strategies for the row player and the column player of M respectively.

Consider

$$\begin{pmatrix} \lambda\boldsymbol{p} & (1-\lambda)\boldsymbol{r} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} \lambda\boldsymbol{p}A & (1-\lambda)\boldsymbol{r}B \end{pmatrix}.$$

By the definition of $\boldsymbol{p}$ and $\boldsymbol{r}$, we have each of the first m coordinates of $(\lambda\boldsymbol{p}A, (1-\lambda)\boldsymbol{r}B)$ is at least λu, and each of the last n coordinates of $(\lambda\boldsymbol{p}A, (1-\lambda)\boldsymbol{r}B)$ is at least $(1-\lambda)v$. Since $u, v > 0$, by letting $\lambda u = (1-\lambda)v$, we have $\lambda = \frac{v}{u+v}$ and $\lambda u = \frac{uv}{u+v}$. Then we have each element of the vector

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \frac{v}{u+v}\boldsymbol{q}^T \\ \frac{u}{u+v}\boldsymbol{s}^T \end{pmatrix} = \begin{pmatrix} \frac{v}{u+v}A\boldsymbol{q}^T \\ \frac{u}{u+v}B\boldsymbol{s}^T \end{pmatrix}$$

is at most $\frac{uv}{u+v}$. More over,

$$\begin{pmatrix} \frac{v}{u+v}\boldsymbol{p} & \frac{u}{u+v}\boldsymbol{r} \end{pmatrix} \begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} \frac{v}{u+v}\boldsymbol{q}^T \\ \frac{u}{u+v}\boldsymbol{s}^T \end{pmatrix} = \frac{uv}{u+v}.$$

Hence by the Minimax Theorem, we have the value of M is $\frac{uv}{u+v}$, $(\frac{v}{u+v}\boldsymbol{p}, \frac{u}{u+v}\boldsymbol{r})$ is an optimal strategy for the row player and $(\frac{v}{u+v}\boldsymbol{q}, \frac{u}{u+v}\boldsymbol{s})$ is an optimal strategy for the column player.